STATISTICAL EQUATIONS OF TURBULENT MOTION IN LAGRANGIAN VARIABLES

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The present paper is concerned with the statistical dynamics of the turbulent motion in Largrangian variables. The set of Lagrangian distribution equations for the coordinates and velocities of fluid particles forms an interlocking system, and to decouple these equations we use the assertion that large-scale and small-scale motions are independent. We obtain a closed equation for the simultaneous probability density of the velocity and coordinate of a single particle. In the homogeneous case, the latter becomes the simultaneous normal distribution of the velocity and coordinate.

The statistical description of phenomena characterizing the turbulent motion of an incompressible fluid necessitates the use of Lagrangian variables and the investigation of the motion of the fluid particles moving in accordance with the hydrodynamic equations. The problem of turbulent diffusion, i.e. of expansion of a passive admixture in the turbulent flow, constitutes such a problem. In a number of cases we can neglect the fact that the admixture may take part in the process of molecular diffusion. When the admixture is assumed to be continuously distributed, the problem of turbulent diffusion will be that of supplying a statistical description of the concentration field n $(\mathbf{X},\ t)$. Such a feature of the concentration field as the probability density F_1 $(n,\ X,\ t)$ that the concentration at the point X and time t is equal to n, is related to the probability density of the distribution P_1 $(\mathbf{X},\ t;\ \alpha,\ t_0)$ of the coordinate X of the fluid particle at the point a when $t = t_0$

$$F_1(n, \mathbf{X}, t) = \int P_1(\mathbf{X}, t; \mathbf{a}, t_0)\, \delta(n - n(\mathbf{a}, t_0))\, d\mathbf{a}$$

In the presence of a steady point source Q at the point a, the mean concentration can also be defined in terms of P_1

$$\langle n(\mathbf{X}, t)\rangle = Q \int_{-\infty}^{t} P_1(\mathbf{X}, t; \mathbf{a}, t_0)\, dt_0$$

Here and in the following the brackets $\langle\ \ \rangle$ denote averaging over an ensemble.

When solving another class of problems, we find that it is not sufficient to consider just P_1 characterizing the motion of a single particle. Such problems include that of the relative diffusion, i.e. of the turbulent diffusion of an admixture cloud described in the terms of P_2 $(\mathbf{X}_1,\ \mathbf{a}_1,\ \mathbf{X}_2,\ \mathbf{a}_2,\ t,\ t_0)$ that is, of the simultaneous probability distribution density of the coordinates $\mathbf{X}_1$ and $\mathbf{X}_2$ of two fluid particles appearing in $\mathbf{a}_1$ and $\mathbf{a}_2$ at $t = t_0$. Dispersion of the admixture cloud is connected with the relative dispersion of two fluid particles in the following manner

$$\sigma = \int (\mathbf{X}_1 - \mathbf{X}_2)^2 P_2 n(\mathbf{a}_1, t_0)\, n(\mathbf{a}_2, t_0)\, d\mathbf{X}_1 d\mathbf{X}_2 d\mathbf{a}_1 d\mathbf{a}_2$$

The coordinates X^α $(\mathbf{a}, t)$ at the instant t of the fluid particles which appear at the points a^β when $t = t_0$, represent the unknowns in Lagrange equations. The matrix

$$\frac{\partial a^\alpha}{\partial X^\beta} = \left| \frac{\partial X^m}{\partial a^n} \right|^{-1} \frac{\partial X^{\alpha_1}}{\partial a^{\beta_1}} \frac{\partial X^{\alpha_2}}{\partial a^{\beta_2}} \varepsilon_{\alpha_1\alpha_2\alpha} \varepsilon_{\beta_1\beta_2\beta}$$

transforms the derivatives with respect to the Eulerian coordinates $\mathbf{X}$, to the derivatives with respect to the initial $\mathbf{a}$.

When the fluid is incompressible, the Jacobian $|\partial X^m / \partial a^n|$ of this variable transformation from X^m to a^n is equal to unity. Hydrodynamic equations for an incompressible viscous fluid, where pressure is expressed in terms of velocity

$$\frac{\partial V^\alpha}{\partial t} + V^\beta \frac{\partial V^\alpha}{\partial X^\beta} = -\int \frac{\partial V^\beta}{\partial X_2{}^\gamma} \frac{\partial V^\gamma}{\partial X_2{}^\beta} G^\alpha(\mathbf{X}_1, \mathbf{X}_2)\, d\mathbf{X}_2 + \nu \Delta V^\alpha$$

in Lagrangian coordinates, become

$$\frac{\partial^2 X^\alpha}{\partial t^2} = -\int \frac{\partial^2 X^\beta}{\partial t \partial a^s} \frac{\partial a^s}{\partial X^\gamma} \frac{\partial^2 X^\gamma}{\partial t \partial a^k} \frac{\partial a^k}{\partial X^\beta} G^\alpha d\mathbf{a}_2 + \nu \frac{\partial a^k}{\partial X^\beta} \frac{\partial}{\partial a^k} \left(\frac{\partial a^l}{\partial X^\beta} \frac{\partial^2 X^\alpha}{\partial t \partial a^l} \right)$$

In expressing the pressure in terms of velocity, use is made of Green's function G appearing in the Neumann problem of the potential theory and defined by $G^\alpha = \partial G / \partial X^\alpha{}_1$.

The statistical description of the turbulent motion, in terms of Lagrangian variables, is based on the set of distribution functions of the coordinates and velocities P_n $(\mathbf{V}_1, \mathbf{X}_1, \mathbf{a}_1, t_1, \ldots, \mathbf{V}_n, \mathbf{X}_n, \mathbf{a}_n, t)$ such that the probability dW, the velocities and coordinates of n fluid particles appearing at the points $\mathbf{a}_1, \ldots, \mathbf{a}_n$ when $t = t_0$, fall within the range $d\mathbf{V}_1 d\mathbf{X}_1 \ldots d\mathbf{V}_n \mathbf{X}_n$ at the instants $t_1, \ldots, t_n$, is $P_n d\mathbf{V}_1 d\mathbf{X}_1 \ldots d\mathbf{V}_n d\mathbf{X}_n$. In the following we shall consider the case $t_1 = t_2 = \ldots = t_n = t$.

The equations of evolution for the function P_n and additional conditions are conveniently derived by representing P_n as the mean over an ensemble of

$$\prod_{i=1}^{n} \delta(\mathbf{X}_i - \mathbf{X}(a_i, t))\, \delta(\mathbf{V}_i - \mathbf{X}^\bullet(\mathbf{a}_i, t))$$

P_n are easily shown to be normalized by

$$\int P_{n+1} d\mathbf{V}_{n+1} dX_{n+1} = P_n, \qquad \int P_1 d\mathbf{V}_1 dX_1 = 1$$

Since the field $X^\alpha(\mathbf{a}, t)$ is continuous $P_{n+1} = P_n \delta(\mathbf{V}_i - \mathbf{V}_k) \delta(\mathbf{X}_i - \mathbf{X}_k)$, when $\mathbf{a}_i = \mathbf{a}_k$. The latter does not, in general, hold for $\nu = 0$ when tangential discontinuities are possible.

The Eulerian velocity field $V^\alpha(\mathbf{X}, t)$ and the Lagrangian trajectories are related by

$$\frac{\partial X^\alpha}{\partial t} = V^\alpha(\mathbf{X}(\mathbf{a}, t), t) \tag{1}$$

When the statistical description of turbulence is given from the Eulerian point of view, the distribution functions F_n $(\mathbf{V}_1, \mathbf{X}_1, \ldots, \mathbf{V}_n, \mathbf{X}_n, t)$ considered [1, 2] represent the probability density of the fact that at the instant t, the flow velocities at the fixed points $\mathbf{X}_1, \ldots, \mathbf{X}_n$ fall within $d\mathbf{V}_1 \ldots d\mathbf{V}_n$. Clearly, the probability of observing the velocity $\mathbf{V}$ at the fixed point $\mathbf{X}$ is defined by the Lagrangian probability summed over all the fluid particles; the velocity at the point $\mathbf{X}$ is equal to $\mathbf{V}$. Indeed, let us integrate P_n over the initial coordinates $\mathbf{a}_1, \ldots, \mathbf{a}_n$

$$\left\langle \int \prod_{i=1}^{n} \delta(\mathbf{V}_i - \mathbf{X}^\bullet(\mathbf{a}_i, t))\, \delta(\mathbf{X}_i - \mathbf{X}(\mathbf{a}_i, t))\, d\mathbf{a}_1 \ldots d\mathbf{a}_n \right\rangle \tag{2}$$

This can be done by changing to new variables $\mathbf{X}(\mathbf{a}_i, t)$, with the Jacobian of this transformation equal to unity. Then using (1), we obtain

$$\left\langle \prod_{i=1}^{n} \delta(\mathbf{V}_i - \mathbf{V}(X_i, t)) \right\rangle$$

Thus

$$E_n = \int P_n(\mathbf{V}_1, X_1, \mathbf{a}_1, \cdots, \mathbf{V}_n, X_n, \mathbf{a}_n, t)\, d\mathbf{a}_1 \cdots d\mathbf{a}_n \quad (3)$$

Integration with respect to the initial coordinates is performed over the volume occupied by the fluid.

In this manner we can construct distribution functions posse.sing m Eulerian and $n-m$ Lagrangian arguments.

A theorem due to Lumly [5] stating that the probability distributions for the Eulerian and Lagrangian velocities coincide when the turbulence in an incompressible fluid is homogeneous, represents a particular case (homogeneous turbulence and $n = 1$) of (3).

Lagrangian distributions have a distinctive property connected with the incompressibility of the fluid, namely since

$$\delta(\mathbf{V}_i - \mathbf{X}^{\cdot}(\mathbf{a}_i, t))\,\delta(\mathbf{X}_i - \mathbf{X}(\mathbf{a}_i, t)) = \delta(\mathbf{a}_i, -\mathbf{a}(\mathbf{X}_i, t))\,\delta(\mathbf{V}_i - \mathbf{V}(\mathbf{X}_i, t))$$

P_n can be regarded of the distribution of not X, but the initial coordinate a. From this it follows that P_n also represents the probability density dW, that the velocity is equal to $\mathbf{V}_i$ at the fixed point $\mathbf{X}_i$, and particles possessing these velocities arrive from random points $\mathbf{a}_i$ with the probability density corresponding to $dW = P_n d\mathbf{V}_1 d\mathbf{a}_1 \ldots d\mathbf{V}_n d\mathbf{a}_n$.

The equations of motion of the initial coordinates $a\ (X,\ t)$ in terms of the final coordinates and time have the form

$$\frac{\partial a^\alpha}{\partial t} + V^\beta(\mathbf{X}, t)\frac{\partial a^\alpha}{\partial X^\beta} = 0, \quad a^\alpha(\mathbf{X}, t_0) = X^\alpha \quad (4)$$

Using this point of view we can represent P_n as a mean over the ensemble of

$$\prod_{i=1}^{n} \delta(\mathbf{V}_i - \mathbf{V}(\mathbf{X}_i, t))\,\delta(\mathbf{a}_i - \mathbf{a}(\mathbf{X}_i, t)) \quad (5)$$

The equations of motion of the random values $V^\alpha\ (\mathbf{X},\ t)$ and $a^\alpha\ (\mathbf{X},\ t)$ consist of the Navier-Stokes equations and the transport equations of the initial coordinates (4). Differentiating (5) with respect to t and utilizing the equations of motion for $V^\alpha\ (\mathbf{X},\ t)$ and $a^\alpha\ (\mathbf{X},\ t)$ we obtain a system of linked equations for P_n in a more compact form that that given in [3]

$$\frac{\partial P_n}{\partial t} + \sum_{i=1}^{n} \frac{\partial P_n}{\partial X_i^\alpha} V_i^\alpha + \frac{\partial}{\partial V_i^\alpha} \int \frac{\partial^2 P_{n+1}}{\partial X_{n+1}{}^\beta \partial X_{n+1}{}^\gamma} V_{n+1}{}^\beta V_{n+1}{}^\gamma G^\alpha(\mathbf{X}_i, \mathbf{X}_{n+1}) \times$$
$$\times d\mathbf{V}_{n+1} d\mathbf{X}_{n+1} d\mathbf{a}_{n+1} + \nu \frac{\partial}{\partial V_i^\alpha} \int \delta(\mathbf{X}_i - \mathbf{X}_{n+1})\,\Delta_{n+1} P_{n+1} V_{n+1}^\alpha d\mathbf{V}_{n+1} d\mathbf{X}_{n+1} d\mathbf{a}_{n+1} = 0 \quad (6)$$

Integration of the latter with respect to the initial coordinates $\mathbf{a}_1, \ldots, \mathbf{a}_n$ yields a chain of equations for F_n, and the condition of incompressibility becomes

$$\int \frac{\partial P_n}{\partial X_i^\alpha} V_i^\alpha d\mathbf{V}_i d\mathbf{a}_i = 0$$

The distinctive character of the dependence of P_n on their arguments $\mathbf{X}_i$ stems from the fact that $\mathbf{X}_i$ does not appear in (5) as a random quantity. In particular, the integral $\int P_n d\mathbf{V}_i d\mathbf{a}_i$ is independent of $\mathbf{X}_i$. The definition of P_n furnishes the following condition expressing this fact

$$\frac{\partial P_n}{\partial X_i^\alpha} = -\frac{\partial}{\partial V_i^\beta} \int \frac{\partial P_{n+1}}{\partial X_{n+1}{}^\alpha} V_{n+1}{}^\beta \delta(\mathbf{X}_i - \mathbf{X}_{n+1})\, d\mathbf{X}_{n+1} d\mathbf{V}_{n+1} d\mathbf{a}_{n+1} -$$
$$- \frac{\partial}{\partial a_i^\beta} \int \frac{\partial P_{n+1}}{\partial X_{n+1}{}^\alpha} a_{n+1}{}^\beta \delta(\mathbf{X}_{n+1} - \mathbf{X}_i)\, d\mathbf{X}_{n+1} d\mathbf{V}_{n+1} d\mathbf{a}_{n+1}$$

This can be replaced by the condition of compatibility at the initial instant t_0. The latter form is preferable, as the Lagrangian problem is essentially temporal. Since

$$P_n = F_n \prod_{i=1}^{n} \delta(\mathbf{X}_i - \mathbf{a}_i) \quad \text{for} \quad t = t_0$$

the condition can be reduced to conditions imposed on the Eulerian functions F_n at the instant $t = t_0$

Existing hypotheses imply that the structure of the developed turbulence at large Reynolds numbers R represent a set of unordered pulsations of various sizes ξ and velocities ω, in which the local velocity gradients $\omega_\xi / \xi \gg U / L$ when $\xi \ll L$ (L and U are the characteristic dimension and velocity of large scale motions). Small scale velocity gradients are bounded in such a manner that $\omega_0\xi_0 / \nu \sim 1$. Since the motions are chaotic and only weak connections exist between the large scale and small scale motions, the statistical mode of the small scale pulsations with $\xi \ll L$ is steady and universal [4] and should not be governed by the specific properties of the large scale motions. It follows that neither $\mathbf{X}$ and $\mathbf{V}$ nor their fluctuations can be used to characterize the small scale components of the motion, as they are basically governed by the large scale motions of the fluid elements.

Following [4] it is expedient to consider the relative motions of the fluid particles, i.e. their motion in relation to some definite particle. We can replace the coordinates $\mathbf{a}_i$, $\mathbf{X}_i$ and velocities V_i of the fluid particles by

$$\begin{aligned} \mathbf{X} &= \mathbf{X}_1, \quad \xi_i = \mathbf{X}_i - \mathbf{X}_1 \\ \mathbf{V} &= \mathbf{V}_1, \quad \omega_i = \mathbf{V}_i - \mathbf{V}_1 \quad (i = 2, 3, \ldots, n) \\ \mathbf{a} &= \mathbf{a}_1, \quad \alpha_i = \mathbf{a}_i - \mathbf{a}_1 \end{aligned}$$

Equations (6) will then become

$$\begin{aligned} &\frac{\partial P_n}{\partial t} + V^\alpha \frac{\partial P_n}{\partial X^\alpha} + \sum_{i=2}^{n} \frac{\partial P_{n+1}}{\partial \xi_i^\alpha} \omega_i^\alpha + \\ &+ \frac{\partial}{\partial V^\alpha} \int \frac{\partial^2 P_{n+1}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma} \omega_{n+1}^\beta \omega_{n+1}^\gamma G(\mathbf{X}, \mathbf{X} + \xi_{n+1})\, d\omega_{n+1} d\xi_{n+1} d\alpha_{n+1} + \\ &+ \nu \frac{\partial}{\partial V^\alpha} \int \Delta_{n+1} P_{n+1} \omega_{n+1}^\alpha \delta(\xi_{n+1})\, d\xi_{n+1} d\omega_{n+1} d\alpha_{n+1} + \\ &+ \frac{\partial}{\partial \omega_i^\alpha} \int \frac{\partial^2 P_{n+1}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma} [G^\alpha(\mathbf{X} + \xi_i, \mathbf{X} + \xi_{n+1}) - G^\alpha(\mathbf{X}, \mathbf{X} + \xi_{n+1})] \times \\ &\times \omega_{n+1}^\beta \omega_{n+1}^\gamma d\omega_{n+1} d\xi_{n+1} d\alpha_{n+1} + \nu \frac{\partial}{\partial \omega_i^\alpha} \int [\delta(\xi_{n+1} - \xi_i) - \delta(\xi_{n+1})]\, \Delta_{n+1} P_{n+1} \times \\ &\times \omega_{n+1}^\alpha d\xi_{n+1} d\alpha_{n+1} = 0 \end{aligned} \qquad (7)$$

The latter equations contain terms of differing order of magnitude. When investigating the properties of P_n over small intervals $\xi_i \ll L$, we may expect that the characteristic velocity difference ω of points at these distances is such, that

$$\omega_i / \xi_i \ll U / L$$

When applied to the equations of the distribution functions, this e.g. means that

$$\omega_i^\alpha \frac{\partial P_n}{\partial \xi_i^\alpha} \gg \frac{\partial P_n}{\partial X^\alpha} V^\alpha$$

$$\frac{\partial}{\partial \omega_i^\alpha} \int \Delta_{n+1} P_{n+1} \left[\delta(\xi_i - \xi_{n+1}) - \delta(\xi_{n+1})\right] \omega_{n+1}^\alpha d\omega_{n+1} d\xi_{n+1} d\alpha_{n+1} \gg$$
$$\gg \frac{\partial}{\partial V^\alpha} \int \Delta_{n+1} P_{n+1} \delta(\xi_{n+1}) \omega_{n+1}^\alpha d\omega_{n+1} d\xi_{n+1} d\alpha_{n+1} \quad \text{etc.}$$

When the Kolmogorov [4] factors $\omega_0 = (\varepsilon\nu)^{1/4}$ and $\xi_0 = (\nu^3/\varepsilon)^{1/4}$ are used to estimate the parameters of the relative motion, the neglected terms are at least as small as $R^{-1/4}$. Inspecting e. g. the equations for P_1, we see that they include P_2 integrated over the difference of the initial coordinates α_2. This function can be found from the system (7) for the semi-Eulerian functions

$$P_n'(\mathbf{V}, \mathbf{X}, \mathbf{a}, \omega_2, \xi_2, \ldots, \omega_n, \xi_n, t) = \int P_n d\alpha_2 \ldots d\alpha_n$$

which are Lagrangian only with respect to a single particle. With the small terms neglected, the equation for P_n becomes

$$\frac{\partial P_n'}{\partial t} + \sum_{i=2}^{n} \frac{\partial P_n'}{\partial \xi_i^\alpha} \omega_i^\alpha +$$
$$+ \frac{\partial}{\partial \omega_i^\alpha} \int \frac{\partial^2 P_{n+1}}{\partial \xi_{n+1}^\beta \partial \xi_{n+1}^\gamma} \frac{\partial}{\partial \xi_{n+1}^\alpha} \left[\frac{1}{|\xi_{n+1} - \xi_i|} - \frac{1}{|\xi_{n+1}|}\right] \omega_{n+1}^\beta \omega_{n+1}^\gamma d\omega_{n+1} d\xi_{n+1} +$$
$$+ \nu \frac{\partial}{\partial \omega_i^\alpha} \int \Delta_{n+1} P_{n+1} \left[\delta(\xi_i - \xi_{n+1}) - \delta(\xi_{n+1})\right] \omega_{n+1}^\alpha d\omega_{n+1} d\xi_{n+1} = 0 \qquad (8)$$

We assume that in accordance with our discussion, P_n' have the form

$$P_n' = P_1(\mathbf{V}, \mathbf{X}, \mathbf{a}, t) \varphi_n^{(0)}(\omega_2, \xi_2, \ldots, \omega_n, \xi_n) \qquad (9)$$

The functions satisfy the stationary system of equations and coincide with the Eulerian distributions of the velocity differences $\omega_2, \ldots, \omega_n$ at the distances $\xi_2, \ldots, \xi_n$. They are homogeneous, isotropic, universal, and are not determined by the distinctive properties of the large scale motions but by the temporal changes of the kinetic energy due to these motions which is converted into heat by viscous friction

$$\varepsilon = \frac{1}{2} \nu \int \delta(\xi_2) \Delta_2 \varphi_2^{(0)} \omega_2^2 d\omega_2 d\xi_2$$

It was proposed in [4] that the Eulerian distribution functions over small distances are of such a character. We note that the hypothesis on the independence of the small and large scale motions of the type (9) can be formulated for all t only for the function P_n'. For P_n, the zeroth approximation similar to (9) has a meaning only for small time intervals $t - t_0 \ll L / U$, since the average particles in a turbulent flow move apart by the distance L in the interval of time equal to L/U, even if their initial separation was small and equal to $\alpha \ll L$. Additional conditions for the universal functions $\varphi_n^{(0)}$ are considered in [1].

It would appear that an attempt to close the equation for P_1 using (9) would yield the closed equation for P_1 together with some universal moments of $\varphi_n^{(0)}$. The assumed character of $\varphi_n^{(0)}$, however, implies that these averages are equal to zero.

From the chain of Eqs. (7) it follows that the zeroth approximation should have a correction of the form $P_n'^{(1)} = \frac{\partial P_1}{\partial V^\alpha} \varphi_{n\alpha}^{(1)}(\omega_2, \xi_2, \ldots, \omega_n, \xi_n)$

where $\varphi_{n\alpha}^{(1)}$ are the same as those given in expansion for F_1 proposed in [1].

We have explained before, that a correction of such a type requires that the symmetry conditions hold to a corresponding accuracy. For the function P_n' the latter has the form

$$P_n' = \int P_n(\mathbf{V} + \omega_i,\ \mathbf{X} + \xi_i,\ \alpha + \alpha_i,\ \omega_2 - \omega_i,\ \xi_2 - \xi_i,\ \alpha_2 - \alpha_i, \ldots \\ \ldots, -\omega_i,\ -\xi_i,\ -\alpha_i, \ldots, \omega_n - \omega_i,\ \xi_n - \xi_i,\ \alpha_n - \alpha_i)\, d\alpha_2 \ldots d\alpha_n \tag{10}$$

The other additional condition can be reduced to symmetry conditions with respect to permutations of the groups of arguments $\omega_i \xi_i \alpha_i$.

Since we consider small values of ξ_i, we assume that the main contribution towards this integral is made by α_i which are very small compared with the basic scale. Expanding the right side of (10) with an accuracy to the first order terms and utilizing the fact that in the terms containing derivatives, the zeroth approximation yields the required accuracy, we obtain the approximate symmetry condition

$$P_n' = \widetilde{P}_n' + \omega_i^\alpha \frac{\partial P_1}{\partial V^\alpha} \varphi_n^{(0)} + \frac{\partial P_1}{\partial a^k} \int \alpha_i^k \varphi_n(\omega_i,\ \xi_i,\ \alpha_i)\, d\alpha_2 \ldots d\alpha_n$$

From this we see that the symmetry condition will be satisfied to the same accuracy only if the following first order correction is introduced

$$\frac{\partial P}{\partial a^\alpha} \chi_{n\alpha}^{(1)}(\omega_2,\ \xi_2, \ldots, \omega_n,\ \xi_n,\ t)$$

For $n = 2$ we find, following the method used in [1], that

$$\varphi_{2\alpha}^{(1)} = \frac{1}{2}\, \omega_2^\alpha \varphi_2^{(0)}(\omega_2,\ \xi_2), \qquad \chi_{2\beta}^{(1)} = \frac{1}{2} \int \alpha_2^\beta \varphi_2(\omega_1,\ \xi_2,\ \alpha_2,\ t)\, d\alpha_2$$

Inserting these corrections into the equation for P_1, we obtain (for simplicity we consider the homogeneous case) the following closed equation for P_1:

$$\frac{\partial P_1}{\partial t} + V^\alpha \frac{\partial P_1}{\partial Y^\alpha} + \mu(t) \frac{\partial^2 P_1}{\partial V^\alpha \partial Y^\alpha} + \frac{\varepsilon}{3} \frac{\partial^2 P_1}{\partial V^\alpha \partial V^\alpha} = 0 \tag{11}$$

where $\mathbf{Y} = \mathbf{X} - \mathbf{a}$, and μ which can be found from the integrals of the universal functions $\chi_{2\alpha}^{(1)}$, depends only on ε, ν and t. It has the dimension of velocity squared and it should be independent of viscosity ν ; hence $\mu = \gamma\varepsilon t$.

Equation (11) has a structure typical of Markov random processes, although the coefficient of diffusion in the velocity space is negative.

It is evident that the solution of (11) satisfying the initial condition $P_1(\mathbf{V},\ \mathbf{Y},\ t_0,\ t_0) = F_1(\mathbf{V},\ t_0)\delta(\mathbf{Y})$ will represent a simultaneous normal distribution for $\mathbf{Y}$ and $\mathbf{V}$. As was explained in [1], in the homogeneous case the distribution $F_1(\mathbf{V},\ t)$ is normal

$$P_1(\mathbf{V},\ \mathbf{Y},\ t) \sim \exp \frac{-\mathbf{Y}^2 \langle \mathbf{V}^2 \rangle + 2\mathbf{Y}\mathbf{V} \langle \mathbf{V}\mathbf{Y} \rangle - \mathbf{V}^2 \langle \mathbf{Y}^2 \rangle}{\langle \mathbf{Y}^2 \rangle \langle \mathbf{V}^2 \rangle - \langle \mathbf{Y}\mathbf{V} \rangle^2}$$

The corresponding averages can easily be found by solving a closed set of second moment equations. The requirement that

$$\langle \mathbf{Y}^2 \rangle \langle \mathbf{V}^2 \rangle - \langle \mathbf{Y}\mathbf{V} \rangle^2 > 0$$

yields the condition for γ. If the law of decay of the homogeneous turbulence is

$$\langle \mathbf{V}^2 \rangle = \langle \mathbf{V}^2(0) \rangle\ (1 + t/T)^{-1}$$

then $\gamma = 2$, and

$$\langle \mathbf{Y}^2 \rangle = \langle \mathbf{V}^2(0) \rangle\, T\, [t - T \ln(1 + t/T)], \quad \langle \mathbf{Y}\mathbf{V} \rangle = \langle \mathbf{V}^2(0) \rangle (1 + t/T)^{-1}$$

Thus, when $t \gg T$, the square of the displacement $\langle \mathbf{Y}^2 \rangle \sim t$ just as in the case of random wanderings.

The other result which emerges from the existing notions on the structure of turbulence takes the form of a hypothesis given in [6]. It states that evolution of the velocity and coordinate increments $\Delta\mathbf{V} = \mathbf{V}(t_0 + \tau) - \mathbf{V}(t_0)$, $\Delta\mathbf{X} = \mathbf{X}(t_0 + \tau) - \mathbf{V}(t_0)\tau - a$ of a fluid particle for $\tau \ll T$ can be represented by a Markov random process in a six-dimensional phase space $\{\Delta\mathbf{X}, \Delta\mathbf{V}\}$, and the coefficient of diffusion in the velocity space is proportional to ε in the corresponding equation. The same point of view with regard to the character of accelerations for $R \gg 1$ was used in [7], which dealt with the moments of the relative motion of two fluid particles.

The present method is directly adjacent to the method of expanding the distribution functions in $R^{1/4}$, developed in [1]. The structure of expansion of the Lagrangian distributions is such that it can be transformed into the previously obtained expansion for F_n, by employing the relation (3) connecting F_n with P_n as explained above. When $t = t_0$, the correction connected with dP_1 / da is of higher order of smallness than the terms present in the expansion, and can therefore be assumed to be zero with accuracy of up to $R^{-1/4}$. It must be noted that Eq. (11) does not contain small parameters and cannot therefore yield the closed equation for the distribution density P_1 (**X**, a, t) of the diffusion type.

BIBLIOGRAPHY

1. Ulinich, F. R. and Liubimov, B. Ia., The statistical theory of turbulence of an incompressible fluid at large Reynolds numbers. ZhETF, Vol. 55, №3, (9), 1968.
2. Ulinich, F. R., Statistical dynamics of the turbulent incompressible fluid. Dokl. Akad. Nauk SSSR, Vol. 183, №3, 1968; Monin, A. S. Equations of turbulent motion, PMM Vol. 31, №6, 1967.
3. Liubimov, B. Ia., Lagrangian description of the dynamics of a turbulent motion. Dokl. Akad. Nauk SSSR, Vol. 184, №5, 1969.
4. Kolmogorov, A. N., Local turbulence structure in a viscous incompressible fluid at very large Reynolds numbers. Dokl. Akad. Nauk SSSR, Vol. 30, №4, 1941.
5. Lumly, I., The mathematical nature of Lagrangian and Eulerian statistical functions. Mécanique de la turbulence, coll. Int. du CNRS à Marceille, Paris, Éd. CNRS, 1962.
6. Obukhov, A. M., Description of the turbulence of Lagrangian variables. Coll. "Atmospheric Diffusion and Air Pollution", M., Izd. inostr. lit., 1962.
7. Lin, C. C., On a theory of dispersion by continuous movements. Proc. Nat. Acad. Sci. USA, Vol. 46, №4, 1960.

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